

Constructive martingale representation using Functional Itô Calculus: a local martingale extension

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Abstract

The constructive martingale representation theorem and the vertical derivative of Functional Itô Calculus are extended, from the space of square integrable martingales, to a space of local martingales. The relevant filtration is the augmented filtration generated by a Wiener process.

Keywords: Functional Itô Calculus, Martingale representation, Vertical derivative

1 Introduction

Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which there lives an n -dimensional Wiener process $W(t), 0 \leq t \leq T$, where $T < \infty$. Let $\underline{\mathcal{F}} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$ denote the augmentation under \mathbb{P} of the filtration generated by W . We remark that it should be taken as implicitly understood that if a process is a martingale, or has some other relevant property, then this is so relative to $(\mathbb{P}, \underline{\mathcal{F}})$. Moreover, all local martingales that we introduce below are to be understood as having RCLL sample paths.

One of the main properties of the Itô integral is that it is a local martingale. One of the main results of Itô calculus is that the reverse implication is also true. This result is known as the martingale representation theorem [11, p. 184] [20, p. 189]: *Let M be a RCLL local martingale, then there exists a progressively measurable n -dimensional process φ such that*

$$\int_0^T |\varphi(t)|^2 dt < \infty, \quad M(t) = M(0) + \int_0^t \varphi(s)' dW(s), \quad 0 \leq t \leq T \text{ a.s.}$$

In particular, M has continuous sample paths a.s. $|\cdot|$ denotes the Euclidean norm and $'$ denotes transpose.

Considerable effort has been made in order to find explicit formulas for the integrand φ , i.e. in order to find constructive representations of (local) martingales. Most of the general research on constructive martingale representation

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has been within Malliavin calculus [10, 12, 15, 16, 22]. Here the constructive martingale representation is based on the Malliavin derivative and is known as the Clark-Haussmann-Ocone formula. The class of martingales to which the Clark-Haussmann-Ocone formula can be applied is, however, limited by the condition that the terminal value of the martingale must be Malliavin differentiable. A constructive representation for the class of square integrable martingales is however possible, when considering a distribution-valued generalization of the Malliavin derivative [1, 18].

Recently a new type of constructive martingale representation has been studied using the recently developed Functional Itô Calculus [2, 5, 6, 7]. In what follows we will give a brief account of the Functional Itô Calculus relevant to martingale representation and the augmented Wiener generated filtration $\underline{\mathcal{F}}$. The main references are [2, 6]. For related remarks see Remark 2.8.

Denote an n -dimensional sample path by ω (think of ω as e.g. a sample path of the Wiener process W). Denote a sample path stopped at t by w_t , i.e. let $w_t(s) = \omega(t \wedge s)$, $0 \leq s \leq T$. We consider real-valued functionals of sample paths $F(t, \omega)$ which are a *non-anticipative* (essentially meaning that $F(t, \omega) = F(t, w_t)$). The *horizontal derivative* at (t, ω) is defined by

$$\mathcal{D}F(t, \omega) = \lim_{h \searrow 0} \frac{F(t+h, w_t) - F(t, w_t)}{h}.$$

The *vertical derivative* at (t, ω) is defined by $\nabla_\omega F(t, \omega) = (\partial_i F(t, \omega), i = 1, \dots, n)'$, where

$$\partial_i F(t, \omega) = \lim_{h \rightarrow 0} \frac{F(t, w_t + h e_i I_{[t, T]}) - F(t, w_t)}{h}.$$

Higher order vertical derivatives are obtained by vertically differentiating vertical derivatives. If the functional F is sufficiently regular (regarding e.g. continuity and boundedness of its derivatives), which we write as $F \in \mathbb{C}_b^{1,2}$ [2, p. 131], then the functional Itô formula holds [2, ch. 6]. The functional Itô formula is the standard Itô formula with the usual derivatives replaced by the horizontal and vertical derivatives.

Using the functional Itô formula it is easy to see that if Z is a martingale satisfying

$$Z(t) = F(t, W_t) \quad dt \times d\mathbb{P}\text{-a.e.}, \text{ with } F \in \mathbb{C}_b^{1,2}, \quad (1)$$

then, for every $t \in [0, T]$,

$$Z(t) = Z(0) + \int_0^t \nabla_\omega F(s, W_s)' dW(s) \quad a.s.$$

We may therefore define the vertical derivative with respect to the process W of a martingale Z satisfying (1) as the $dt \times d\mathbb{P}$ -a.e. unique process $\nabla_W Z$ given by

$$\nabla_W Z(t) = \nabla_\omega F(t, W_t), \quad 0 \leq t \leq T. \quad (2)$$

Let $\mathcal{C}_b^{1,2}(W)$ be the space of processes Z which allow the representation in (1). Let $\mathcal{L}^2(W)$ be the space of progressively measurable processes φ with $E[\int_0^T \varphi(s)' \varphi(s) ds] < \infty$. Let $\mathcal{M}^2(W)$ be the space of square integrable martingales with initial value 0. Let $D(W) = \mathcal{C}_b^{1,2}(W) \cap \mathcal{M}^2(W)$.

It turns out that $\{\nabla_W Z : Z \in D(W)\}$ is dense in $\mathcal{L}^2(W)$ and that $D(W)$ is dense in $\mathcal{M}^2(W)$. Using this it is possible to show that the vertical derivative

(operator) $\nabla_W(\cdot)$ admits a unique extension to $\mathcal{M}^2(W)$: For $Y \in \mathcal{M}^2(W)$ the (weak) vertical derivative $\nabla_W Y$ is the unique element in $\mathcal{L}^2(W)$ satisfying

$$E[Y(T)Z(T)] = E\left[\int_0^T \nabla_W Y(t)' \nabla_W Z(t) dt\right] \quad (3)$$

for every $Z \in D(W)$ (where $\nabla_W Z$ is defined in (2)). Using the above it is possible to prove the following general constructive martingale representation theorem [2, p. 171]: For any square integrable martingale Y relative to $(\mathbb{P}, \underline{\mathcal{F}})$ and every $t \in [0, T]$,

$$Y(t) = Y(0) + \int_0^t \nabla_W Y(s)' dW(s) \text{ a.s.} \quad (4)$$

The present paper contains:

- An extension of the vertical derivative $\nabla_W(\cdot)$ to local martingales, see Theorem 2.2 and Definition 2.4.
- A constructive representation theorem for local martingales, see Theorem 2.5.

Remark 1.1 *Many of the applications using martingale representation are in mathematical finance. A particular application that would benefit from the constructive martingale representation in Theorem 2.5 is optimal investment theory, in which the discounted (using the state price density) optimal wealth process is a (not necessarily square integrable) martingale [12, ch. 3]. In particular, using Theorem 2.5 it would, under certain general conditions, be possible to derive an explicit formula for the optimal portfolio in terms of the vertical derivative of the discounted optimal wealth process. Similar explicit formulas for optimal portfolios based on the Malliavin calculus approach to constructive martingale representation have, under more restrictive assumptions, been studied extensively, see e.g. [4, 8, 9, 13, 14, 17, 18, 19, 21].*

2 Constructive representation of local martingales

The current definition of the vertical derivative $\nabla_W(\cdot)$, cf (3), relies crucially on the square integrability of the process that is being differentiated. We will start by extending the definition of the vertical derivative to local martingales with initial value zero; by defining this vertical derivative as the limit of the vertical derivatives (using the current definition) of the local martingale suitable stopped. Let us first recall the following definition.

Definition 2.1 *M is said to be a local martingale if there exists a sequence of non-decreasing stopping times $\{\theta_n\}$ with $\lim_{n \rightarrow \infty} \theta_n = \infty$ a.s., such that the stopped local martingale $M(\cdot \wedge \theta_n)$ is a martingale for each $n \geq 1$ [11, p. 36].¹*

¹Note that infinite-valued stopping times are possible in the present setting since the infimum of the empty set is infinity by convention, cf [23].

Let $\mathcal{M}^{\text{loc}}(W)$ denote the space of local martingales relative to $(\mathbb{P}, \underline{\mathcal{F}})$ with initial value zero and RCLL sample paths.

Theorem 2.2 (Definition of $\nabla_W(\cdot)$ on $\mathcal{M}^{\text{loc}}(W)$)

- *There exists a progressively measurable $dt \times d\mathbb{P}$ -a.e. unique extension of the vertical derivative $\nabla_W(\cdot)$ from $\mathcal{M}^2(W)$ to $\mathcal{M}^{\text{loc}}(W)$, such that, for $M \in \mathcal{M}^{\text{loc}}(W)$,*

$$\int_0^T |\nabla_W M(t)|^2 dt < \infty, \quad M(t) = \int_0^t \nabla_W M(s)' dW(s), \quad 0 \leq t \leq T \quad \text{a.s.} \quad (5)$$

- *Specifically, for $M \in \mathcal{M}^{\text{loc}}(W)$ the vertical derivative $\nabla_W M$ is defined as the progressively measurable $dt \times d\mathbb{P}$ -a.e. unique process satisfying*

$$\nabla_W M(t) = \lim_{n \rightarrow \infty} \nabla_W M_n(t) \quad dt \times d\mathbb{P}\text{-a.e.} \quad (6)$$

where $\nabla_W M_n$ is the vertical derivative of $M_n := M(\cdot \wedge \tau_n) \in \mathcal{M}^2(W)$ and τ_n is given by

$$\tau_n = \theta_n \wedge \inf \{s \in [0, T] : |M(s)| \geq n\} \wedge T \quad (7)$$

where $\{\theta_n\}$ is an arbitrary sequence of stopping times of the kind described in Definition 2.1.

Remark 2.3 *If M in Theorem 2.2 satisfies $M(t) = \int_0^t \gamma(s)' dW(s)$, $0 \leq t \leq T$ a.s. for some process γ , then $\gamma = \nabla_W M \, dt \times d\mathbb{P}$ -a.e. [11, p. 182-184]. It follows that the extended vertical derivative $\nabla_W M$ defined in Theorem 2.2 does not depend (modulo possibly on a null set $dt \times d\mathbb{P}$) on the particulars of the chosen stopping time τ_n . All we need from τ_n is that $M_n := M(\cdot \wedge \tau_n)$ is a square integrable martingale, as the proof below reveals.*

Proof. The martingale representation theorem implies that there, for $M \in \mathcal{M}^{\text{loc}}(W)$, exists a progressively measurable process φ satisfying

$$\int_0^T |\varphi(t)|^2 dt < \infty, \quad M(t) = \int_0^t \varphi(s)' dW(s), \quad 0 \leq t \leq T, \quad \text{a.s.} \quad (8)$$

Therefore, if we can prove that

$$\lim_{n \rightarrow \infty} \nabla_W M_n(t) = \varphi(t) \quad dt \times d\mathbb{P}\text{-a.e.}, \quad (9)$$

then it follows that there exists a progressively measurable process, denote it by $\nabla_W M$, which is $dt \times d\mathbb{P}$ -a.e. uniquely defined by (6) and satisfies $\nabla_W M(t) = \varphi(t) \, dt \times d\mathbb{P}$ -a.e., which in turn implies that the integrals of $\nabla_W M$ and φ coincide in the way that (8) implies (5). All we have to do is therefore to prove that (9) holds.

Let us recall some results about stopping times and martingales. The stopped local martingale $M(\cdot \wedge \theta_n)$ is a martingale for each n , by Definition 2.1. The minimum of two stopping times is a stopping time and the hitting

time $\inf \{s \in [0, T] : |M(s)| \geq n\}$ is, for each n , a stopping time [3], [11, p. 46,5]. Stopped RCLL martingales are martingales [11, p. 20]. Using this it follows that

$$M(\cdot \wedge \theta_n \wedge \inf \{s \in [0, T] : |M(s)| \geq n\} \wedge T) = M(\cdot \wedge \tau_n)$$

is a martingale, for each n . Moreover, M is by the standard martingale representation a.s. continuous. Hence, we may define a sequence of, a.s. continuous, martingales $\{M_n\}$ by

$$M_n = M(\cdot \wedge \tau_n) = \int_0^{\cdot \wedge \tau_n} \varphi(s)' dW(s) \text{ a.s.} \quad (10)$$

where the last equality follows from (8). Now, use the definition of τ_n in (7) to see that

$$|M_n(t)| = \left| \int_0^{t \wedge \tau_n} \varphi(s)' dW(s) \right| \leq n \text{ a.s.}$$

for any t and n , and that in particular M_n is, for each n , a square integrable martingale. Moreover, (10) implies [11, p. 140,147] that M_n satisfies

$$M_n(t) = \int_0^t I_{\{s \leq \tau_n\}} \varphi(s)' dW(s), \quad 0 \leq t \leq T \text{ a.s.} \quad (11)$$

We may now, since each M_n is a square integrable martingale, use the current constructive martingale representation theorem of Functional Itô Calculus (cf. (4)) on M_n , which together with (11) implies that

$$M_n(t) = \int_0^t \nabla_W M_n(s)' dW(s) = \int_0^t I_{\{\tau_n \leq s\}} \varphi(s)' dW(s), \quad 0 \leq t \leq T \text{ a.s.} \quad (12)$$

where $\nabla_W M_n$ is the vertical derivative of M_n with respect to W (defined in (3)) and where we also used the continuity of the Itô integrals. The equality of the two Itô integrals in (12) implies [11, p. 182] that

$$\nabla_W M_n(t) = I_{\{t \leq \tau_n\}} \varphi(t) \quad dt \times d\mathbb{P}\text{-a.e.} \quad (13)$$

The local martingale property of M implies that $\lim_{n \rightarrow \infty} \theta_n = \infty$ a.s. Using this and the definition of τ_n in (7) we conclude that for almost every $\omega \in \Omega$ and each $t \in [0, T]$ there exists an $N(\omega, t)$ such that

$$n \geq N(\omega, t) \Rightarrow \sup_{0 \leq s \leq t} |M(\omega, s)| \leq n \text{ and } t \leq \theta_n(\omega) \Rightarrow t \leq \tau_n(\omega) \Rightarrow t \wedge \tau_n = t. \quad (14)$$

Hence, it follows from (13) and (14) that there exists an $N(\omega, t)$ such that

$$n \geq N(\omega, t) \Rightarrow \nabla_W M_n(\omega, t) = \varphi(\omega, t) \quad dt \times d\mathbb{P}\text{-a.e.}$$

which means that (9) holds. ■

Clearly, if M is a RCLL local martingale then $M - M(0) \in \mathcal{M}^{\text{loc}}(W)$, which implies that $\nabla_W(M - M(0))$ is defined in Theorem 2.2. Let us use this observation to extend the definition of the vertical derivative to RCLL local martingales not necessarily starting at zero.

Definition 2.4 *The vertical derivative of a local martingale M relative to $(\mathbb{P}, \underline{\mathcal{F}})$ with RCLL sample paths is defined as the progressively measurable $dt \times d\mathbb{P}$ -a.e. unique process $\nabla_W M$ satisfying*

$$\nabla_W M(t) = \nabla_W(M - M(0))(t), \quad 0 \leq t \leq T, \quad (15)$$

with $\nabla_W(M - M(0))$ defined in Theorem 2.2.

The following result is an immediate consequence of Theorem 2.2 and Definition 2.4.

Theorem 2.5 (Main result) *If M is a local martingale relative to $(\mathbb{P}, \underline{\mathcal{F}})$ with RCLL sample paths, then*

$$\int_0^T |\nabla_W M(t)|^2 dt < \infty, \quad M(t) = M(0) + \int_0^t \nabla_W M(s)' dW(s), \quad 0 \leq t \leq T, \quad a.s.$$

Relationship with the Malliavin calculus approach to constructive martingale representation

The following is a general version of the constructive martingale representation theorem of Malliavin calculus, known as the Clark-Haussmann-Ocone formula [10, 12]: *For every \mathcal{F}_T -measurable and once Malliavin differentiable random variable $F \in \mathbf{D}_{1,1}$,*

$$F = E[F] + \int_0^T E_{\mathcal{F}_t}[(D_t F)'] dW(t) \quad a.s.$$

The integrand $E_{\mathcal{F}_t}[(D_t F)'], 0 \leq t \leq T$ is the predictable projection of the Malliavin derivative of F . For a short description of the spaces $\mathbf{D}_{p,k}$ see [10]. Furthermore, the integrand process φ of the standard martingale representation of the martingale $E_{\mathcal{F}_t}[F], 0 \leq t \leq T$ satisfies [10, p. 5]

$$\varphi(t) = E_{\mathcal{F}_t}[D_t F] \quad dt \times d\mathbb{P}\text{-a.e.} \quad (16)$$

Theorem 2.5 and (16) (see also (6), (9) and (15)) imply the following relationship between the vertical derivative and the Malliavin derivative.

Corollary 2.6 *If M is a martingale relative to $(\mathbb{P}, \underline{\mathcal{F}})$ with RCLL sample paths and $M(T) \in \mathbf{D}_{1,1}$, then*

$$\nabla_W M(t) = \nabla_W E_{\mathcal{F}_t}[M(T)] = E_{\mathcal{F}_t}[D_t M(T)], \quad dt \times d\mathbb{P}\text{-a.e.}$$

Remark 2.7 *A similar result was established for $\mathbf{D}_{2,1} \subset \mathbf{D}_{1,1}$ in [2, 6] (where $\mathbf{D}_{2,1}$ is denoted by $\mathbf{D}^{1,2}$).*

Remark 2.8 *Let us comment on some of the results of Functional Itô Calculus related to the present paper. The constructive martingale representation result that we present in (4) is originally presented in a setting where our Wiener process W is replaced by a Brownian martingale, see [2, ch. 7.2]. A constructive representation of "smooth local martingales" is available in [2, p. 167]; smooth local martingales are local martingales satisfying certain restrictive regularity conditions, including a localized version of the representation in (1). The vertical derivative can be extended to square integrable semimartingales [2, ch. 7.5].*

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